GEOMETRICAL ANALYSIS OF NON-LINEAR CURVATURE TRANSITION CURVES OF HIGH SPEED RAILWAYS

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ABSTRACT

The transition curves in the modern railway construction are route elements equally crucial as alignment and tracking. For high speed routes, in order to prevent a sudden change of the centrifugal force, the nonlinear curvature transition curve must be applied due to the impact of the motion in a sharp curve. This article proposes a new curve representing transition curve as a polynomial non linear curvature transition curve which is considered as a remodeling of the third degree parabolic non – linear transition curve. Formulation of the curve relies on a pre – valuation of the lateral shock such that it increases gradually to the maximum value at the midpoint, then decreases gradually and tangent to x – axis at both the start and end points. Various transition curves adjusted for the high speed railways tracks, including the proposed one, are geometrically examined for the purpose of studying the possibility of considering such transition curves in high speed railways network. The concerned curves are the new and these needed to be optimized, which are designed for high speed up to 350 km/h. Based on the enhancement of the effect of the dynamical elements, i. e. lateral shock, curvature and lateral displacement, the comparative study cited from the geometrical analysis assured that he second degree parabolic non – linear and the proposed transition curve are the most relevant curves to be connected to the railways routes.

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1. INTRODUCTION

The selection of a suitable transition curve is of major importance towards a proper alignment and design in railway projects. Trains in operation are exposed to the risk of derailment resulting from severe vibration induced by centrifugal forces if the following three situations occur: (1) the train enters a curve from a straight line; (2) the train travels from a curve to another curve of different radius; (3) the train leaves a curve and enters a straight line. These sudden changes can also cause discomfort to passengers. Hence, transition curves must be placed between two circular curves and where straight lines meet circular curves [1].

The transition curve is characterized by its curvature as function of the longitudinal position. For controlled (low) speed, linear curvature transition curves are aligned. For high speed trains of 350 km/hr, nonlinear curvature transition curve are applied to overcome the lateral shock by allowing gradual change in curvature and lateral acceleration [2].

For the purpose of accomplishing the optimal and the recommended transition curve, geometric analysis for various transition curves is illustrated addressing the factors related to the high speed railways dynamics which are curvature and the rate of change of the lateral acceleration with respect to time, i. e. lateral shock [3]. These curves are the second degree parabolic non linear curvature transition curve, the third degree parabolic non linear curvature transition curve, the hyperbolic tangent non linear curvature transition curve and the inverse tangent non linear curvature transition curve. In addition, a new transition curve as a polynomial non linear curvature is proposed, which is considered as a remodeling of the second curve, the third degree parabolic non linear transition curve. Formulation of such transition curve relies on a pre valuation of the lateral shock such that it increases gradually to the maximum value at the midpoint, then decreases gradually.

2. Transition Curves Geometry

Curvature $k$ of a curve in cartesian coordinates system is defined as [1]:

$$k = \frac{\rho}{y} \quad (1)$$

$$k = \frac{y}{\left(1 + y^2 \right)^{3/2}} \quad (2)$$

where $\rho$ is the radius of curvature at the concerned point on the curve. The curve angle $\phi_x$ is the angle between the tangent at the start point of the curve, $x \text{ axis}$ and the tangent at a given point $P$ of coordinates $(x, y)$ (Figure 1).
Such angle may be determined by integrating curvature equation, Eq. 1, as:
\[ \varphi_x = \int k_x \, dx \]  \hspace{1cm} (3)

Equation of the curve may be obtained by integrating the curve angle equation, Eq. 2, as:
\[ y = \int \varphi_x \, dx = \int k_x \, d^2x \]  \hspace{1cm} (4)

Variation of the lateral acceleration, lateral shock, \( \psi \) is defined as the rate of change of the lateral acceleration with respect to the time, i.e.:
\[ \psi = \frac{db}{dt} \]  \hspace{1cm} (5)

where the maximum value of the lateral shock is 1.0 \( \text{m/sec}^3 \), and \( b \) is the lateral acceleration whose value ranging between 0.55 \( \text{m/sec}^3 \) to 0.6 \( \text{m/sec}^3 \), and:
\[ \psi = \frac{v^2}{Rl} \]  \hspace{1cm} (6)

\( v \) is the velocity, \( R \) is the radius of the circular curve and \( l \) is the length of the transition curve.

3. Linear Curvature Transition Curve

Linear curvature transition curves (Figure 2) are used in horizontal design of railways routes of controlled speeds, up to 120 km/h and usually they are of linear curvature [2]. Linear curvature may be represented as:
\[ k_x = \frac{k}{l} \, x = \frac{1}{IR} \, x \]  \hspace{1cm} (7)

And by integrating equation (7), curve angle is:
\[ \varphi_x = \frac{k}{2l} \, x^2 = \frac{1}{2IR} \, x^2 \]  \hspace{1cm} (8)

and by integrating equation (8), curve equation is:
\[ y_x = \frac{k}{6l} \, x^3 = \frac{1}{6IR} \, x^3 \]  \hspace{1cm} (9)

Equation (9) is a cubic parabola, which is considered as a familiar transition curve for designing railways routes [4].

Also, another familiar linear curvature transition curves are the Clothoids [4], whose points are of coordinates:
\[ x = s \left( 1 - \frac{s^4}{40(Rl)^2} + \frac{s^6}{2456(Rl)^4} - \frac{s^{12}}{599040(Rl)^6} \right) \]  \hspace{1cm} (10)

\[ y = s \left( \frac{s^2}{6lR} - \frac{s^6}{366(Rl)^3} - \frac{s^{12}}{42240(Rl)^5} \right) \]  \hspace{1cm} (11)

Where \( s \) is the distance measured along the curve from its start point. It has to be noted that the cubic parabola curve matches clothoid curve at radii of curvature greater than 3000 m [4]. Figure 5 shows that the lateral shock is constant along the whole length of the linear curvature transition curve, and equals zero for circular curves. As shown in the figure, lateral shock value jumps from infinity to a certain value at the start point, and jumps to zero value at the end point.

4. Non-Linear Curvature Transition Curves

Non-linear curvature transition curves are used when designing high speed railways, more than 160 km/hr [3]. Elements of transition curves must be evaluated from
the point of view of influencing trains dynamic conditions when entering circular curves. Concerning this issue, additional element has been introduced and evaluated, which is the lateral shock that arose according to the centrifugal force occurrence. Such element has to be evaluated as acceptance / rejection of the non–linear curvature transition curves. Various non–linear transition curves, including linear curvature transition curve of equation (9), were analytically analyzed as follows.

4.1. Second Degree Parabolic Non–Linear Curvature Transition Curve

Equation of the second degree parabolic non–linear curvature transition curve is:

\[ F(x) = ax^2 + bx + c \]  \hspace{1cm} (12)

According to Figure 6, the first half of the curvature curve is a positive parabola whose vertex is the origin of the coordinates system, and its curvature equation is:

\[ k_{11} = \frac{-2k}{l^2} x^2, \hspace{0.5cm} 0 \leq x \leq l \]  \hspace{1cm} (13)

The second half of the curvature curve is a negative parabola whose vertex is point 2 at the end of the curve, length equals \( L \), at which curvature equals \( 1/R \) and its equation is:

\[ k_{12} = \frac{-2k}{l^2} x^2 + \frac{4k}{l} x - k, \hspace{0.5cm} \frac{1}{2} \leq x \leq l \]  \hspace{1cm} (14)

Integrating equations (13) and (14) gives curve angle, as:

\[ \varphi_{11} = \frac{2}{3Rl^2} x \]  \hspace{1cm} (15)

\[ \varphi_{12} = -\frac{2}{3Rl^2} x^3 + \frac{2}{Rl} x^2 - \frac{1}{R} x + \frac{1}{6R} \]  \hspace{1cm} (16)

And the integration of equations (15) and (16) gives equation of the transition curve, as:

\[ y_{11} = \frac{1}{6Rl^2} x^4 \]  \hspace{1cm} (17)

\[ y_{12} = -\frac{1}{6Rl^2} x^4 + \frac{2}{3Rl} x^3 - \frac{1}{2R} x^2 + \frac{1}{6R} x - \frac{l^2}{48R} \]  \hspace{1cm} (18)

The lateral shock curve (Figure 7) is linearly increases from zero to the maximum value at the mid–point, then linearly decreases from the maximum value to zero at the end point. Lateral shock may be obtained mathematically by differentiating curvature equations, equations (13) and (14), as:

\[ \psi_{11} = \frac{4k}{l^2} x \]  \hspace{1cm} (19)

\[ \psi_{12} = \frac{-4k}{l^2} x + \frac{4k}{l} \]  \hspace{1cm} (20)

4.2. Third Degree Parabolic Non–Linear Curvature Transition Curve

Equation of the third degree parabolic non–linear Transition Curve is:

\[ F(x) = ax^3 + bx^2 + cx + d \]  \hspace{1cm} (12)

Such transition curve is authorized by German Railways, whose curvature curve is cubic parabola. On the curvature curve (Figure 8), the midpoint is the reverse point at \( 1/2R \), which the curvature equals. In a similar way to the previous case, equations of curvature, angle, and the curve are:

\[ k_2 = \frac{2}{Rl^2} x^2 - \frac{2}{Rl^2} x^3 \]  \hspace{1cm} (21)

\[ \varphi_2 = \frac{1}{R} x^3 - \frac{1}{2Rl^2} x^4 \]  \hspace{1cm} (22)

\[ y_2 = \frac{1}{4Rl^2} x^4 - \frac{1}{10Rl^2} x^5 \]  \hspace{1cm} (23)

In Figure 9, the lateral shock values non–linearly increases from zero to the maximum value at the mid–point then non–linearly decreases to zero value at the end of the transition curve. Such values can be obtained by differentiating the curvature equation, as:

\[ \psi_2 = \frac{6}{Rl^2} x - \frac{6}{Rl^3} x^2 \]  \hspace{1cm} (24)
4.3. Hyperbolic Tangent Non-Linear Curvature Transition Curve

Equation of the hyperbolic tangent non-linear transition curve is:

$$F(x) = \tanh(x)$$  \hfill (25)

Where x-axis is the curve asymptote at the start point and the line \( k = 1/R \) is the curve asymptote at the end point (Figure 10).

The reverse point of the curvature curve has the coordinates \((L/2, 1/2R)\). Equation of such curve is:

$$X = \left( x - \frac{L}{2} \right) \frac{2n}{l}$$

$$k_4 = \frac{1}{2R} \arctan \left( \frac{x - L/2}{\pi} \right)$$  \hfill (29)

Figure 10: Hyperbolic Tangent Non-Linear Curvature Transition Curve.

The accompanied lateral shock curve's equation is (Figure 11):

$$\psi_3 = \left( 1 - \tanh^2 x \right) \frac{\pi}{R_l}$$  \hfill (27)

Figure 11: Lateral Shock of Hyperbolic Tangent Non-Linear Curvature Transition Curve.

4.4. Inverse Tangent Non-Linear Curvature Transition Curve

Equation of the inverse tangent non-linear transition curve is \([5, 6, 7]\):

$$F(x) = \arctan \left( \frac{x}{k} \right)$$  \hfill (28)

where x-axis is the curve asymptote at the start point and the line \( k = 1/R \) is the curve asymptote at the end point (Figure 12).

The reverse point of the curvature curve has the coordinates \((L/2, 1/2R)\). Equation of such curve is:

$$X = \left( x - \frac{L}{2} \right) \frac{2n}{l}$$

$$k_4 = \frac{1}{2R} \arctan \left( \frac{x - L/2}{\pi} \right)$$

Figure 12: Curvature of the Inverse Tangent Non-Linear Curvature Transition Curve.

The accompanied lateral shock curve's equation is (Figure 13):

$$\psi_4 = \frac{\pi}{2R L (1 + x^2)}$$  \hfill (30)

Figure 13: Lateral Shock of the Inverse Tangent Non-Linear Curvature Transition Curve.

4.5. Polynomial Non-Linear Curvature Transition Curve

This paper proposes a new curve representing transition curve as a polynomial non-linear curvature transition curve which is considered as a remodeling of the second curve, the third degree parabolic non-linear transition curve. Formulation of the curve relies on a pre-valuation of the lateral shock such that it increases gradually to the maximum value at the midpoint, then decreases gradually. Unlike the third degree parabolic non-linear transition curve whose lateral shock curve forms acute angles with x-axis, the lateral shock curve of the polynomial non-linear curvature transition curve is tangent x-axis at both the start and end points (Figure 14). To accomplish the curve's equation, firstly the lateral shock smooth curve is proposed as:
Integrating lateral shock equations gives the curvature equations, where:

\[
\psi = \frac{8 \Psi_m}{l^2} x^2; \quad 0 \leq x \leq \frac{l}{4}
\]

(31)

\[
\psi = \psi_m - \frac{8 \Psi_m}{l^2} \left( x - \frac{l}{2} \right)^2; \quad \frac{l}{4} < x \leq \frac{3l}{4}
\]

(32)

\[
\psi = \frac{8 \Psi_m}{l^2} (l - x)^2; \quad \frac{3l}{4} < x \leq l
\]

(33)

Where the lateral shock curve is the differentiation of the curvature curve, then the area under the lateral shock curve equals, i.e. \(1/R\). Therefore, the maximum value of the lateral shock, at the midpoint, is determined such that the area under the lateral shock for the linear case equals the area under the lateral shock curve for the polynomial case (Figure 14).

5. Comparative Analysis of Transition Curves

In this section, a comparative study for the previous various transition curves is introduced in order to illustrate the relevant curves for the high speed railways used and to be used in Egypt. The study based on values of:

\[L = 150 \text{ m}\]

\[R = 3000 \text{ m}\]

which are consistent with Egyptian railways considerations [8].

Values of lateral shock, curvature, angles and coordinates were calculated along the length of the curve for points at 20 m apart. Such values are shown in the following figures.

The lateral shock curve behaves according to the transition curve type [3]. For the cubic parabola curvature transition curve, the lateral shock is constant along the entire length of the transition curve, equals infinity along the straight track, and equals zero at the beginning of the circular curve, i.e. changes as jumps.

For the non-linear second degree polynomial curvature transition curve, the first model, the lateral shock values increases linearly from zero to reaches the maximum value at the midpoint of the curve, then decreases linearly from the maximum value to reaches the zero value at end point of the curve.

Concerning the third degree parabolic non-linear curvature transition curve, the second model, the lateral shock curve is parabolic shape, forms ample angles with \(x\)– axis and its maximum value at the midpoint.

The hyperbolic tangent non-linear curvature transition curve, the third model, lateral shock curve is steeply changes to reach the maximum value, more than triple values of the linear model, at the midpoint.

For the inverse tangent non-linear curvature transition curve, the fourth model, the lateral shock curve changes from the start point by a value of 25% of its midpoint maximum value, then decreases to the same value at the end point.

Concerning the proposed transition curve, the polynomial non-linear curvature transition curve, the lateral shock curve changes smoothly, tangent to \(x\)– axis at the
start point, and increases to its maximum value at the midpoint then decreases in the same way to reach the end point.
The maximum value of the lateral shock for this model equals the double value of the corresponding value for the linear model.

6. Results and Recommendations

Figure 16 shows the relation between the lateral shock of the six different types of transition curves presented, where the subscript 0 in $\psi_0$ denotes the linear curvature transition curve; subscript 1 in $\psi_1$ denotes the second degree parabolic non-linear curvature transition curve; subscript 2 in $\psi_2$ denotes the third degree parabolic non-linear transition curve, subscript 3 in $\psi_3$ denotes the hyperbolic tangent non-linear transition curve; subscript 4 in $\psi_4$ denotes the inverse tangent non-linear transition curve, and subscript 5 in $\psi_5$ denotes the polynomial non-linear curvature transition curve.
The figure prove that the transition curves are so close to each other along the first half of the transition curves, and diverges as reaching the end point.

![Figure 16: Lateral Shocks of the Six Transition Curve Models.](image)

![Figure 17: Curve Radius and Length Versus Lateral Shock.](image)

Figure 17: Curve Radius and Length Versus Lateral Shock.

Both the radius and length of the curves affect the maximum value of the lateral shock. For example, for the polynomial non-linear curvature transition curve, and for the favor of attaining lateral shock value less than $1.0 \text{ m/sec}^2$, at the velocity of value $250 \text{ km/hr}$, transition curve's length equals $200 \text{ m}$ and radius equals $3000 \text{ m}$, or for transition curve's length of $150 \text{ m}$ and radius equals $4000 \text{ m}$ (Figure 17).
The five non-linear curvature transition curves may be evaluated considering the values of the lateral shock, curvature and lateral displacement as evaluation criteria (Figure 16, 18, 19, 20).

![Figure 18: Curvature of the Six Transition Curve Models.](image)

![Figure 19: Curve Angle of the Six Transition Curve Models.](image)

![Figure 20: Vertical Displacements of the Six Transition Curve Models.](image)

![Figure 21: Lateral Displacements of the Six Transition Curve Models.](image)

The six models were arranged (Table 1) and each curve's criterion was graded a numerical value from one to five [9].

![Table 1: Evaluation of the Non-Linear Curvature Transition Curves.](image)

Table 1: Evaluation of the Non-Linear Curvature Transition Curves.

<table>
<thead>
<tr>
<th>Non-Linear Transition Curve</th>
<th>Evaluation Criterion</th>
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<td>Lateral Shock</td>
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<td>Second Degree Parabolic</td>
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</tr>
<tr>
<td>Third Degree Parabolic</td>
<td>3</td>
</tr>
<tr>
<td>Hyperbolic Tangent</td>
<td>2</td>
</tr>
<tr>
<td>Inverse Tangent</td>
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<td>5</td>
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According to Table 1, it is recommended to consider the...
first and fifth models, the second degree parabolic non-linear and the proposed modeled which is the polynomial curvature transition curves, in the Egyptian high-speed railway track design.

REFERENCES